

On the Quasi-Stationary Distribution of SIS Models

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Abstract

In this paper we propose a novel method for constructing upper bounds of the quasi-stationary distribution of SIS processes. Using this method, we obtain an upper bound that is better than the state-of-the-art upper bound. Moreover, we prove that the fixed point map Φ [7] actually preserves the equilibrium reversed hazard rate order under a certain condition. This allows us to further improve the upper bound. Some numerical results are presented to illustrate the results.

Keywords Birth-death process; equilibrium distribution; likelihood ratio order; stochastic order; total positivity.

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1 Introduction

Consider a birth-death process $\{X(t), t \geq 0\}$ on finite state space $\{0, 1, \dots, N\}$, with birth rate λ_i and death rate μ_i at state i , where $i = 0, \dots, N$, $\lambda_i > 0$ for $i = 1, \dots, N - 1$, $\mu_i > 0$ for $i = 1, \dots, N$, and $\lambda_0 = \lambda_N = \mu_0 = 0$. Note that 0 is the absorbing state. For this class of processes, it is well known that extinction will eventually happen regardless of the initial distribution (i.e., the stationary distribution degenerates to state 0 with probability 1). However, the time to extinction can be very long [32], which prompts to study the long-term behavior before extinction. A popular measurement for this purpose is called *Quasi-Stationary Distribution* (QSD), which has been extensively studied in the literature due to its considerable practical importance in biology, chemical and applied probability models. Specifically, a distribution $\mathbf{q} = (q_1, \dots, q_N)$ is called QSD of $X(t)$ if $P(X(0) = i) = q_i$ for $i = 1, \dots, N$ implies $P(X(t) = i | X(t) > 0) = q_i$ for all $t > 0$. Equivalently, \mathbf{q} is the unique limiting conditional probability distribution with

$$q_i = \lim_{t \rightarrow \infty} P(X(t) = i | X(t) > 0),$$

which justifies why QSD is a good measurement of the long-term behavior prior to the eventual extinction [6]. It is hard to obtain closed-form solutions to QSD, except for a few special cases [23]. This explains why researchers instead resort to stochastic bounds [12, 15, 14] and approximations [29, 19, 3, 31] of QSD.

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On the other hand, Susceptible-Infected-Susceptible (SIS) is a popular model for describing epidemic spreading. This model is widely used in fields including biology, sociology and chemistry [1, 26, 2, 31], and has inspired many studies in the field of cyber security [13, 27, 32, 33, 34, 35]. Many SIS models consider a closed population of N individuals, where each individual only has two possible states at any point in time: *susceptible* and *infected*, where an infected individual is also infectious. A susceptible individual can be infected by an infectious individual according to a Poisson process with rate λ . An infected individual can return to the susceptible state according to a Poisson process with rate μ . The infection events and the recovery events are assumed to be independent of each other.

The SIS model is, in fact, a special case of the birth-death process. Specifically, let $I(t)$ denote the number of infected individuals at time t in the SIS model. Then, $\{I(t), t \geq 0\}$ is exactly the birth-death process $\{X(t), t \geq 0\}$ with birth and death rates

$$\lambda_i = \lambda i(N - i) \quad \text{and} \quad \mu_i = \mu i \quad (1)$$

at state i for $i = 0, \dots, N$, respectively. This means that the concept of QSD mentioned above also applies to the SIS model. Since it is hard to derive explicit solutions to QSD, there have been many studies on bounding and approximating QSD [2, 16, 21, 22, 24, 5, 4, 25]. A popular method for bounding and approximating QSD is to use auxiliary processes that have no absorbing state 0. Cavender [2] introduced an auxiliary process by assuming the death rate at population size 1 is 0 (i.e., $\mu_1 = 0$), and proved that the stationary distribution of this variant process is a lower bound of the QSD \mathbf{q} of the process $\{X(t), t \geq 0\}$.

Along the same direction, Kryscio and Lefèvre [16] introduced another auxiliary process by assuming the existence of a permanently infected individual, namely that the variant process has the same birth rates $\lambda_i = \lambda i(N - i)$ for $i = 1, \dots, N$ as process $\{X(t), t \geq 0\}$, but has instead death rates $\mu_i = \mu(i - 1)$ for $i = 1, \dots, N$. Let $\hat{\mathbf{q}}$ denote the stationary distribution of the variant process. Kryscio and Lefèvre observed empirically that $\hat{\mathbf{q}}$ is an upper bound of \mathbf{q} , i.e.,

$$\mathbf{q} \leq_{\text{st}} \hat{\mathbf{q}}, \quad (2)$$

where \leq_{st} is the usual stochastic order. Clancy and Pollett [5] later proved this upper bound, by utilizing the map Φ that was introduced in [7, 21]. The map Φ is defined as follows. For a given initial distribution $\mathbf{v} = (v_1, \dots, v_N)$, suppose $X(t)$ restarts at state i with probability v_i for $i = 1, \dots, N$ whenever the process hits the absorbing state 0. The restarted process has a finite, irreducible space, and therefore a unique stationary distribution denoted by $\boldsymbol{\rho} = (\rho_1, \dots, \rho_N)$. The map Φ is defined as $\Phi(\mathbf{v}) = \boldsymbol{\rho}$. It is known that $\lim_{n \rightarrow \infty} \Phi^n(\mathbf{v}) = \mathbf{q}$ for any \mathbf{v} and $\Phi(\mathbf{q}) = \mathbf{q}$, where Φ^n denotes the n -composition of Φ , namely that \mathbf{q} is the unique fixed point of the map Φ [7]. It is also known that for process $\{X(t), t \geq 0\}$, Φ is defined as

$$\lambda_{i-1}\rho_{i-1} - (\lambda_i + \mu_i)\rho_i + \mu_{i+1}\rho_{i+1} = -\mu_1\rho_1v_i, \quad i = 1, \dots, N \quad (3)$$

where $\rho_0 = \rho_{N+1} = 0$ [5]. Clancy and Pollett [5] proved upper bound (2) by showing that Φ preserves the likelihood ratio order, namely that the likelihood ratio order in \mathbf{v} is preserved by $\Phi(\mathbf{v}) = \boldsymbol{\rho}$.

In this paper, we make two contributions. First, we propose a new method for bounding QSD from above. This method allows us to obtain a new upper bound that is better than the state-of-the-art upper bound $\hat{\mathbf{q}}$ given in (2), where “better” is in terms of the likelihood ratio order. Second, we identify a sufficient condition under which the transformation Φ actually has the nicer property of preserving the equilibrium reversed hazard rate order. This nicer property allows us to further improve the upper bound of QSD.

The rest of paper is organized as follows. In Section 2, we briefly review some notions and present some lemmas. In Section 3, we describe our new method and construct a new upper bound of QSD. In Section 4, we specify a sufficient condition for stochastically comparing $\Phi(\mathbf{v})$ in terms of hazard rate order, and explore the new preservation property. In Section 5, we use the newly identified preservation property to further improve the upper bound we obtained in Section 3. In Section 6, we conclude the present paper and discuss further research directions.

2 Preliminaries

In this section, we recall some notions and present some lemmas which will be used in the sequel.

Let X and Y be two discrete random variables with support $\mathcal{D} = \{1, 2, \dots, N\}$, where $N \in \mathbb{N}^+$. Denote the probability mass functions of X and Y by $f_i = P(X = i)$ and $g_i = P(Y = i)$, the distribution functions $F_i = \sum_{j=1}^i f_j$ and $G_i = \sum_{j=1}^i g_j$, the survival functions $\bar{F}_i = \sum_{j=i}^N f_j$ and $\bar{G}_i = \sum_{j=i}^N g_j$, respectively. Let $\mathbf{f} = (f_1, \dots, f_N)$ and $\mathbf{g} = (g_1, \dots, g_N)$ represent the probability mass vectors of X and Y , respectively.

Definition 1 Random variable X is said to be less than random variable Y in

- (a) the *usual stochastic order* (denoted by $X \leq_{\text{st}} Y$ or $\mathbf{f} \leq_{\text{st}} \mathbf{g}$) if $\sum_{j=i}^N f_j \leq \sum_{j=i}^N g_j$ for all $i \in \mathcal{D}$;
- (b) the *hazard rate order* (denoted by $X \leq_{\text{hr}} Y$ or $\mathbf{f} \leq_{\text{hr}} \mathbf{g}$) if $\bar{F}_i \bar{G}_j \geq \bar{F}_j \bar{G}_i$ for all $i, j \in \mathcal{D}$ with $i \leq j$;
- (c) the *reversed hazard rate order* (denoted by $X \leq_{\text{rh}} Y$ or $\mathbf{f} \leq_{\text{rh}} \mathbf{g}$) if $F_i G_j \geq F_j G_i$ for all $i, j \in \mathcal{D}$ with $i \leq j$;
- (d) the *likelihood ratio order* (denoted by $X \leq_{\text{lr}} Y$ or $\mathbf{f} \leq_{\text{lr}} \mathbf{g}$) if $f_i g_j \geq f_j g_i$ for all $i, j \in \mathcal{D}$ with $i \leq j$.

It is well known that

$$\mathbf{f} \leq_{\text{lr}} \mathbf{g} \implies \mathbf{f} \leq_{\text{hr(rh)}} \mathbf{g} \implies \mathbf{f} \leq_{\text{st}} \mathbf{g}.$$

We refer to Shaked and Shanthikumar [30] for details about those stochastic orders.

Recall that the equilibrium distribution of a discrete random variable X is defined as

$$P(X_e \leq i) = \frac{1}{\sum_{j=1}^N \bar{F}_j} \sum_{j=1}^i \bar{F}_j, \quad i \in \mathcal{D},$$

where X_e represents the equilibrium version of X . The equilibrium distribution is also known as the stationary renewal function, which has been widely used in the areas of reliability theory, stochastic processes, and maintenance polices etc. [10, 18]. In Section 5, comparing two equilibrium random variables in the sense of reversed hazard rate order will be utilized to prove the main results. To facilitate the discussion there, let us define the reversed hazard rate order between equilibrium random variables as the *equilibrium reversed hazard rate order*.

Definition 2 Let X_e and Y_e be the equilibrium versions of random variables X and Y , respectively. X is said to be less than Y in the *equilibrium reversed hazard rate order* (denoted by $X \leq_{\text{erh}} Y$ or $\mathbf{f} \leq_{\text{erh}} \mathbf{g}$) if $X_e \leq_{\text{rh}} Y_e$, i.e.,

$$\sum_{k=1}^i \bar{F}_k \sum_{m=1}^j \bar{G}_m \geq \sum_{k=1}^j \bar{F}_k \sum_{m=1}^i \bar{G}_m$$

for all $i, j \in \mathcal{D}$ with $i \leq j$.

The following lemma shows that the equilibrium reversed hazard rate order is weaker than the hazard rate order but stronger than the usual stochastic order.

Lemma 1 Let X and Y be two discrete random variables. Then

$$X \leq_{\text{hr}} Y \implies X \leq_{\text{erh}} Y \implies X \leq_{\text{st}} Y. \quad (4)$$

Proof If $X \leq_{\text{hr}} Y$, then for all $m \leq k$, $\bar{F}_m \bar{G}_k \geq \bar{F}_k \bar{G}_m$. Hence, for all $i < j$,

$$\sum_{m=1}^i \bar{F}_m \sum_{k=i+1}^j \bar{G}_k \geq \sum_{m=1}^i \bar{G}_m \sum_{k=i+1}^j \bar{F}_k,$$

which, in turn, implies that

$$\sum_{m=1}^i \bar{F}_m \sum_{k=1}^j \bar{G}_k \geq \sum_{m=1}^i \bar{G}_m \sum_{k=1}^j \bar{F}_k,$$

i.e., $X \leq_{\text{erh}} Y$.

If $X \leq_{\text{erh}} Y$, then it holds that for $j = 2, \dots, N$,

$$\sum_{k=1}^j \bar{G}_k \sum_{m=1}^{j-1} \bar{F}_m \geq \sum_{k=1}^j \bar{F}_k \sum_{m=1}^{j-1} \bar{G}_m,$$

and hence

$$\sum_{k=1}^j \bar{G}_k \sum_{m=1}^j \bar{F}_m - \sum_{k=1}^j \bar{G}_k \sum_{m=1}^{j-1} \bar{F}_m \leq \sum_{k=1}^j \bar{G}_k \sum_{m=1}^j \bar{F}_m - \sum_{k=1}^j \bar{F}_k \sum_{m=1}^{j-1} \bar{G}_m,$$

or

$$\bar{F}_j \sum_{k=1}^j \bar{G}_k \leq \bar{G}_j \sum_{m=1}^j \bar{F}_m.$$

Note that for all j ,

$$\frac{\sum_{k=1}^j \bar{G}_k}{\sum_{m=1}^j \bar{F}_m} \geq \frac{\sum_{k=1}^1 \bar{G}_k}{\sum_{m=1}^1 \bar{F}_m} = 1,$$

then $\bar{F}_j \leq \bar{G}_j$ for all $1 \leq j \leq N$, i.e., $X \leq_{\text{st}} Y$.

The proof is completed. ■

It is interesting to point out that the continuous version of the equilibrium reversed hazard rate order has been explored in [18]; for other related equilibrium orders, one may refer to [10].

We will also use the concept of *Totally Positivity of order 2* (TP₂) of functions.

Definition 3 ([11]) A function $h(x, y)$ is said to be TP_2 if $h(x, y) \geq 0$ and

$$h(x_1, y_1)h(x_2, y_2) \geq h(x_1, y_2)h(x_2, y_1)$$

for all $x_1 \leq x_2$ and $y_1 \leq y_2$.

Lemma 2 ([11]) Let A, B and C be subsets of the real line. Let $L(x, z)$ be TP_2 in $(x, z) \in A \times B$ and $M(z, y)$ be TP_2 in $(z, y) \in B \times C$. Then, $K(x, y) = \int L(x, z)M(z, y) d\mu(z)$ is also TP_2 in $(x, y) \in A \times C$, where μ is a σ -finite measure.

The following characterization of reversed hazard rate order will also be used in the sequel.

Lemma 3 ([30]) Let X and Y be the random variables defined above. Then, $X \leq_{\text{rh}} Y$ implies that $E[\alpha(X)]E[\beta(Y)] \geq E[\alpha(Y)]E[\beta(X)]$, i.e.,

$$\sum_{i=1}^N \alpha(i)f_i \cdot \sum_{j=1}^N \beta(j)g_j \geq \sum_{i=1}^N \alpha(i)g_i \cdot \sum_{j=1}^N \beta(j)f_j$$

for all functions α and β where β is nonnegative, and α/β and β are decreasing.

3 New Upper Bound of QSD

In this section, we present a novel method for constructing upper bound of QSD. Recall that the upper bound (2) is constructed by assuming that there exists a permanently infected individual in the SIS process. Note that our study will be based on the variant SIS model proposed in [9] and further analyzed in [20]. The variant model, called ε -SIS, generalizes the SIS model (1), by assuming that each susceptible individual is subject to self-infection with rate $\varepsilon > 0$, which is in addition to the infection caused by the infected individuals. Note that ε -SIS degenerates to the standard SIS model when $\varepsilon = 0$. It is worth mentioning that the idea of ε is equivalent to the notion of ‘‘pull-based attacks’’ in the setting of cyber epidemics, which was introduced by Li et al. [17] and thoroughly investigated in [33].

Recall also that $I(t)$ is the number of infected individuals at time t in the SIS model. Similarly, we can let $I_\varepsilon(t)$ denote the number of infected individuals in the ε -SIS model reviewed above. It is clear that $I_\varepsilon(t)$ is still a birth-death process, but its birth and death rates are respectively

$$\lambda_{n,\varepsilon} = (\lambda n + \varepsilon)(N - n) \quad \text{and} \quad \mu_n = n\mu,$$

at state n , $n = 0, 1, \dots, N$. Note that unlike $I(t)$ in the SIS model, $I_\varepsilon(t)$ in the ε -SIS model has no absorbing state when $\varepsilon > 0$. This suggests the possibility to use the ε -SIS model with small ε to bound the QSD of the SIS model, in the same fashion as how the aforementioned auxiliary processes with no absorbing states were used to characterize the QSD in the presence of an absorbing state [2, 16, 21, 22, 24, 5, 4, 25].

Denote by $\mathbf{p}_\varepsilon = (p_{0,\varepsilon}, p_{1,\varepsilon}, \dots, p_{N,\varepsilon})$ the stationary distribution of $I_\varepsilon(t)$. According to [28], we have

$$p_{i,\varepsilon} = p_{0,\varepsilon} \pi_{i,\varepsilon}, \quad i = 0, 1, \dots, N,$$

where $p_{0,\varepsilon} = 1 / \sum_{k=0}^N \pi_{k,\varepsilon}$, $\pi_{0,\varepsilon} = 1$, and

$$\pi_{i,\varepsilon} = \frac{\lambda_{0,\varepsilon} \cdots \lambda_{i-1,\varepsilon}}{\mu_1 \cdots \mu_i}, \quad i = 1, \dots, N.$$

Let us define

$$\tau = \lambda/\mu, \quad \varepsilon^* = \varepsilon/\mu,$$

where τ is called *effective infection rate* and ε^* is called *self-infection rate*. Now, $\pi_{i,\varepsilon}$ can be expressed as

$$\pi_{i,\varepsilon^*} = \binom{N}{i} \prod_{k=0}^{i-1} (\varepsilon^* + k\tau), \quad i = 1, \dots, N. \quad (5)$$

The conditional stationary distribution of $I_\varepsilon(t)$ or the QSD of the ε -SIS model, denoted by $\tilde{\mathbf{q}}_{\varepsilon^*} = (\tilde{q}_{1,\varepsilon^*}, \dots, \tilde{q}_{N,\varepsilon^*})$, is given as

$$\tilde{q}_{i,\varepsilon^*} = \lim_{t \rightarrow \infty} \text{P}(I_\varepsilon(t) = i | I_\varepsilon(t) > 0) = \frac{p_{i,\varepsilon}}{1 - p_{0,\varepsilon}}, \quad i = 1, \dots, N.$$

In what follows we show how to use $\tilde{\mathbf{q}}_{\varepsilon^*}$ to construct upper bound for the QSD of the SIS model \mathbf{q} . From (5), for a given τ , $\tilde{q}_{i,\varepsilon^*}$ can be represented as

$$\tilde{q}_{i,\varepsilon^*} = \frac{\pi_{i,\varepsilon^*}}{\sum_{j=1}^N \pi_{j,\varepsilon^*}} = \frac{\binom{N}{i} \prod_{k=0}^{i-1} (\varepsilon^* + k\tau)}{\sum_{j=1}^N \binom{N}{j} \prod_{k=0}^{j-1} (\varepsilon^* + k\tau)} \quad (6)$$

for $i = 1, \dots, N$.

First, we prove that $\tilde{\mathbf{q}}_{\varepsilon^*}$ is increasing in $\varepsilon^* > 0$ in terms of the likelihood ratio order.

Lemma 4 In the ε -SIS model, if $0 < \varepsilon_1^* \leq \varepsilon_2^*$, then

$$\tilde{\mathbf{q}}_{\varepsilon_1^*} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon_2^*}.$$

Proof For $0 < \varepsilon_1^* \leq \varepsilon_2^*$, from (6) it follows that

$$\frac{\tilde{q}_{i,\varepsilon_2^*}}{\tilde{q}_{i,\varepsilon_1^*}} = \delta \prod_{k=0}^{i-1} \frac{(\varepsilon_2^* + k\tau)}{(\varepsilon_1^* + k\tau)},$$

where δ is a positive scalar independent of i . It is easy to see that the ratio is increasing in $1 \leq i \leq N$, which means $\tilde{\mathbf{q}}_{\varepsilon_1^*} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon_2^*}$. ■

Next, we determine the range of ε^* in which $\tilde{\mathbf{q}}_{\varepsilon^*}$ is an upper bound of the QSD of SIS model \mathbf{q} in terms of the likelihood ratio rate order.

Lemma 5 For the QSD of SIS model (1), it holds that for all $\varepsilon^* \geq \varepsilon_+^*$,

$$\mathbf{q} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon^*},$$

where

$$\varepsilon_+^* = \frac{\sqrt{(\tau + 1)^2 + 4\tau} - \tau - 1}{2}. \quad (7)$$

Proof The idea of proof is described as follows. We first prove for all $\varepsilon^* \geq \varepsilon_+^*$, it holds that $\tilde{\mathbf{q}}_{\varepsilon^*} \leq_{\text{lr}} \tilde{\mathbf{v}}_{\varepsilon^*}$, where $\tilde{\mathbf{v}}_{\varepsilon^*} = (\tilde{v}_{1,\varepsilon^*}, \dots, \tilde{v}_{N,\varepsilon^*})$ can be regarded as the pseudo initial distribution of a restarted process $\{I(t), t \geq 0\}$ with stationary distribution $\tilde{\mathbf{q}}_{\varepsilon^*}$ satisfying $\Phi(\tilde{\mathbf{v}}_{\varepsilon^*}) = \tilde{\mathbf{q}}_{\varepsilon^*}$. Then, the preservation property of map Φ on the likelihood ratio order proved by Theorem 1 of [5] is utilized to get the desired result.

From (3), it follows that for $i = 1, \dots, N$,

$$\begin{aligned} \tilde{v}_{i,\varepsilon^*} &= \frac{\lambda_{i-1}\tilde{q}_{i-1,\varepsilon^*} - (\lambda_i + \mu_i)\tilde{q}_{i,\varepsilon^*} + \mu_{i+1}\tilde{q}_{i+1,\varepsilon^*}}{-\mu_1\tilde{q}_{1,\varepsilon^*}} \\ &= \frac{\lambda_{i-1}\pi_{i-1,\varepsilon^*} - (\lambda_i + \mu_i)\pi_{i,\varepsilon^*} + \mu_{i+1}\pi_{i+1,\varepsilon^*}}{-\mu_1\pi_{1,\varepsilon^*}} \\ &= \frac{(\lambda_{i-1,\varepsilon} - \varepsilon(N - i + 1))\pi_{i-1,\varepsilon^*} - (\lambda_{i,\varepsilon} - \varepsilon(N - i) + \mu_i)\pi_{i,\varepsilon^*} + \mu_{i+1}\pi_{i+1,\varepsilon^*}}{-\mu_1\pi_{1,\varepsilon^*}} \\ &= \varepsilon^* \frac{(N - i + 1)\pi_{i-1,\varepsilon^*} - (N - i)\pi_{i,\varepsilon^*}}{\pi_{1,\varepsilon^*}} - \frac{\Delta_i}{\pi_{1,\varepsilon^*}\mu_1} \end{aligned}$$

with

$$\Delta_i = \lambda_{i-1,\varepsilon}\pi_{i-1,\varepsilon^*} - (\lambda_{i,\varepsilon} + \mu_i)\pi_{i,\varepsilon^*} + \mu_{i+1}\pi_{i+1,\varepsilon^*}.$$

Note that for $\varepsilon > 0$, $\{I_\varepsilon(t), t \geq 0\}$ has a finite and irreducible state space. This means that for $i = 1, \dots, N$, the stationary distribution \mathbf{p} of $I_\varepsilon(t)$ must satisfy

$$\lambda_{i-1,\varepsilon}p_{i-1,\varepsilon} - (\lambda_{i,\varepsilon} + \mu_i)p_{i,\varepsilon} + \mu_{i+1}p_{i+1,\varepsilon} = 0,$$

which implies

$$\Delta_i = \lambda_{i-1,\varepsilon}\pi_{i-1,\varepsilon^*} - (\lambda_{i,\varepsilon} + \mu_i)\pi_{i,\varepsilon^*} + \mu_{i+1}\pi_{i+1,\varepsilon^*} = 0.$$

Hence, for $i = 1, \dots, N$, we have

$$\tilde{v}_{i,\varepsilon^*} = \varepsilon^* \frac{(N - i + 1)\pi_{i-1,\varepsilon^*} - (N - i)\pi_{i,\varepsilon^*}}{\pi_{1,\varepsilon^*}}. \quad (8)$$

In the following, we consider two scenarios of $\tilde{\mathbf{v}}_{\varepsilon^*}$.

(i) $\tilde{v}_{i,\varepsilon^*} \geq 0$ for $i = 1, \dots, N$. Then $\tilde{\mathbf{v}}_{\varepsilon^*}$ is a distribution (note that $\sum_{i=1}^N \tilde{v}_{i,\varepsilon^*} = 1$ always holds). Let us prove that for any $\varepsilon^* \geq \varepsilon_+^*$, $\tilde{\mathbf{q}}_{\varepsilon^*} \leq_{\text{lr}} \tilde{\mathbf{v}}_{\varepsilon^*}$, i.e., that for all $1 \leq i < j \leq N$,

$$\tilde{q}_{i,\varepsilon^*}\tilde{v}_{j,\varepsilon^*} - \tilde{q}_{j,\varepsilon^*}\tilde{v}_{i,\varepsilon^*} \geq 0. \quad (9)$$

Equivalently, (9) is rewritten as

$$\pi_{i,\varepsilon^*}[(N - j + 1)\pi_{j-1,\varepsilon^*} - (N - j)\pi_{j,\varepsilon^*}] - \pi_{j,\varepsilon^*}[(N - i + 1)\pi_{i-1,\varepsilon^*} - (N - i)\pi_{i,\varepsilon^*}] \geq 0,$$

namely,

$$(N - j + 1)\frac{\pi_{j-1,\varepsilon^*}}{\pi_{j,\varepsilon^*}} - (N - i + 1)\frac{\pi_{i-1,\varepsilon^*}}{\pi_{i,\varepsilon^*}} + j - i \geq 0. \quad (10)$$

By (5), we have

$$\frac{\pi_{j-1,\varepsilon^*}}{\pi_{j,\varepsilon^*}} = \frac{j}{(N - j + 1)(\varepsilon^* + (j - 1)\tau)}, \quad j = 1, \dots, N,$$

which can be substituted into (10) to obtain

$$[\tau(i-1) + \varepsilon^*][\tau(j-1) + \varepsilon^*] \geq \tau - \varepsilon^*. \quad (11)$$

Since the left-hand side of (11) is increasing in i and j , it is sufficient to prove (11) for the case of $i = 1$ and $j = 2$, i.e.,

$$\varepsilon^*(\tau + \varepsilon^*) \geq \tau - \varepsilon^*.$$

Solving the above inequality yields

$$\varepsilon^* \geq \frac{\sqrt{(\tau+1)^2 + 4\tau} - \tau - 1}{2} \equiv \varepsilon_+^*.$$

Therefore, for all $\varepsilon^* \geq \varepsilon_+^*$, (9) holds.

Since $\Phi(\tilde{\mathbf{v}}_{\varepsilon^*}) = \tilde{\mathbf{q}}_{\varepsilon^*}$, (9) means that for all $\varepsilon^* \geq \varepsilon_+^*$, $\Phi(\tilde{\mathbf{v}}_{\varepsilon^*}) \leq_{\text{lr}} \tilde{\mathbf{v}}_{\varepsilon^*}$. Using the preservation property of map Φ on the likelihood ratio order ([5], or see (15)), we have

$$\Phi^n(\tilde{\mathbf{v}}_{\varepsilon^*}) \leq_{\text{lr}} \Phi^{n-1}(\tilde{\mathbf{v}}_{\varepsilon^*}) \leq_{\text{lr}} \cdots \leq_{\text{lr}} \Phi(\tilde{\mathbf{v}}_{\varepsilon^*}) = \tilde{\mathbf{q}}_{\varepsilon_+^*}.$$

Note that for any \mathbf{v} , we mentioned that

$$\Phi^n(\mathbf{v}) \rightarrow \mathbf{q}, \quad n \rightarrow \infty.$$

Hence, for all $\varepsilon^* \geq \varepsilon_+^*$, $\mathbf{q} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon^*}$.

(ii) $\tilde{v}_{i,\varepsilon^*} < 0$ for some i . The inequality (9) still holds for this case, which is sufficient for the proof of Theorem 1 of [5] to remain valid. That is, we still have the preservation property of likelihood ratio order for this case. Therefore, we conclude that $\mathbf{q} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon^*}$.

The proof is completed. ■

Remark: Negative values of $\tilde{v}_{i,\varepsilon^*}$'s imply that $\tilde{\mathbf{v}}_{\varepsilon^*}$ may not be a distribution. However, this does not impact the result of Lemma 5. The similar issue has been well addressed in Section 2 of [5].

From Lemmas 4 and 5, we immediately get the following upper bound for \mathbf{q} in terms of the likelihood ratio order.

Theorem 1 For the QSD of the SIS model (1), it holds

$$\mathbf{q} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon_+^*} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon^*},$$

where ε_+^* is defined in (7).

Since the likelihood ratio order is stronger than the usual stochastic order, we have:

Corollary 1 For the QSD of the SIS model (1), it holds for all $\varepsilon^* \geq \varepsilon_+^*$,

$$\mathbf{q} \leq_{\text{st}} \tilde{\mathbf{q}}_{\varepsilon_+^*} \leq_{\text{st}} \tilde{\mathbf{q}}_{\varepsilon^*},$$

where ε_+^* is defined in (7).

Now we show that the newly obtained upper bound $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ is better than the state-of-the-art upper bound $\hat{\mathbf{q}}$ given by (2). According to [5], the upper bound $\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2, \dots, \hat{q}_N)$ is given by

$$\hat{q}_i = \frac{(N-1)!}{(N-i)!} \tau^{i-1} \hat{q}_1, \quad i = 1, \dots, N. \quad (12)$$

Note that

$$\frac{\hat{q}_i}{\tilde{q}_{i, \varepsilon_+^*}} \propto \frac{i! \tau^i}{\prod_{k=0}^{i-1} (\varepsilon_+^* + k\tau)} = \frac{\prod_{k=0}^{i-1} (k+1)\tau}{\prod_{k=0}^{i-1} (\varepsilon_+^* + k\tau)} = \prod_{k=0}^{i-1} \frac{(k+1)\tau}{\varepsilon_+^* + k\tau},$$

and $\varepsilon_+^* \leq \tau$. It holds that for all $k = 0, \dots, N-1$,

$$\frac{(k+1)\tau}{\varepsilon_+^* + k\tau} \geq 1.$$

Therefore, $\hat{q}_i / \tilde{q}_{i, \varepsilon_+^*}$ is increasing in $i = 1, \dots, N$ and

$$\mathbf{q} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon_+^*} \leq_{\text{lr}} \hat{\mathbf{q}}. \quad (13)$$

That is, the new upper bound $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ is better than the state-of-the-art upper bound $\hat{\mathbf{q}}$ in terms of the likelihood ratio order and hence the usual stochastic order. This is also confirmed by the following numerical example.

Example 1 Consider the SIS model (1) with $\tau = 0.15$. From (7), $\varepsilon_+^* = 0.1183$. Table 1 presents the numerical results of the QSD \mathbf{q} (evaluated by iterating map Φ), the upper bounds $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ given by (6) and the $\hat{\mathbf{q}}$ given by (12) with $N = 10$. We observe that the ratios $q_i / \tilde{q}_{i, \varepsilon_+^*}$ and $\tilde{q}_{i, \varepsilon_+^*} / \hat{q}_i$ are both decreasing in $i = 1, \dots, 10$, which confirms the result in (13). Therefore, the new upper bound $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ is better than the state-of-the-art upper bound $\hat{\mathbf{q}}$.

To better understand the performance of the bound $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ for population size, we consider the fraction of infected individuals of a population in the quasi-stationary state, i.e., the ratio of the mean of \mathbf{q} to N . From (13), we can get two upper bounds for the ratio from $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ and $\hat{\mathbf{q}}$. Figure 1 displays the fractions of infected individuals of the populations with different sizes in the quasi-stationary state and their upper bounds from $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ and $\hat{\mathbf{q}}$. It is seen that the upper bounds (red curve) based on $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ are better than the ones from $\hat{\mathbf{q}}$ (blue curve) as expected, which coincides with (13). ■

We note that the aforementioned upper bounds $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ and $\hat{\mathbf{q}}$ are in terms of the likelihood ratio order, because the key idea is that map Φ preserves the likelihood ratio order. This motivates us to ask the following question: Does map Φ preserve some weaker stochastic order? If so, can we exploit the weak stochastic order to get an even better upper bound? The following two sections answer these two questions affirmatively.

4 New Preservation Property of Map Φ

In this section, we investigate the preservation property of map Φ in the context of birth-death processes. Recall the birth-death process $X(t)$ described in Section 1. It is known [5] that from (3), ρ can be further expressed as

$$\rho_i = \rho_1 \frac{\mu_1}{\mu_i} \sum_{k=1}^i b_{ik} \bar{v}_k, \quad i = 1, \dots, N, \quad (14)$$

i	q_i	$\tilde{q}_{\varepsilon_+, i}$	\hat{q}_i	$q_i/\tilde{q}_{\varepsilon_+, i}$	$\tilde{q}_{\varepsilon_+, i}/\hat{q}_i$
1	.173	.131	.106	1.321	1.236
2	.187	.157	.143	1.191	1.098
3	.187	.176	.171	1.063	1.029
4	.167	.175	.180	.954	.972
5	.130	.151	.162	.861	.932
6	.086	.109	.121	.789	.901
7	.046	.064	.073	.719	.877
8	.018	.028	.033	.643	.848
9	5.060×10^{-3}	8.149×10^{-3}	9.836×10^{-3}	.625	.829
10	6.951×10^{-4}	1.197×10^{-3}	1.475×10^{-3}	.581	.812

Table 1: Comparing the numerical results of the QSD q , the newly obtained upper bound $\tilde{q}_{\varepsilon_+}$ and the state-of-the-art upper bound \hat{q} in the SIS model, where $\tau = .15$ and $N = 10$.

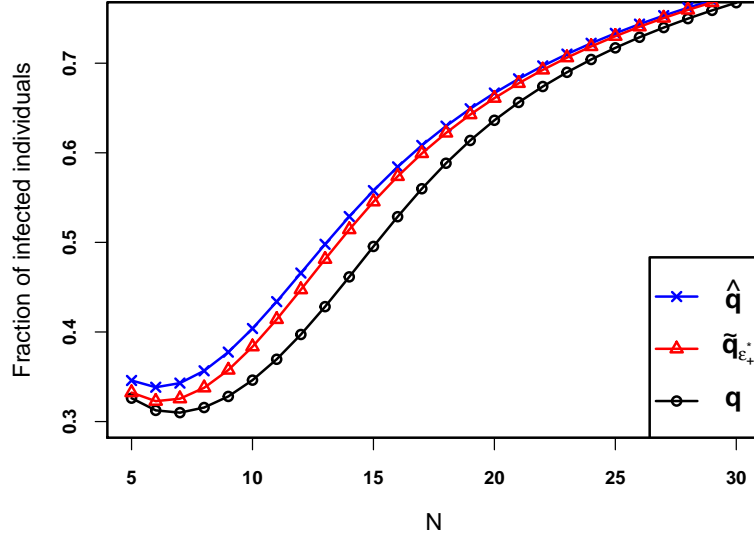


Figure 1: The fractions of infected individuals of the populations with different sizes in the quasi-stationary state and their upper bounds corresponding to \hat{q} and $\tilde{q}_{\varepsilon_+}$, respectively, with $\tau = .15$.

where $\bar{v}_k = \sum_{j=k}^N v_j$, $b_{ik} = \prod_{l=k}^{i-1} \frac{\lambda_l}{\mu_l}$ with $b_{ii} = 1$, and

$$\rho_1^{-1} = \mu_1 \sum_{i=1}^N \sum_{k=1}^i \frac{b_{ik}}{\mu_i} \bar{v}_k.$$

Substituting ρ_1 into (14), we have

$$\rho_i = \frac{\mu_i^{-1} \sum_{k=1}^i b_{ik} \bar{v}_k}{\sum_{i=1}^N \mu_i^{-1} \sum_{k=1}^i b_{ik} \bar{v}_k}$$

for $i = 1, \dots, N$. By utilizing the above expression, [5] showed that for any two initial distributions $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ on $\{1, \dots, N\}$, it holds that

$$\mathbf{v}^{(1)} \leq_{\text{lr}} \mathbf{v}^{(2)} \implies \boldsymbol{\rho}^{(1)} \leq_{\text{lr}} \boldsymbol{\rho}^{(2)}, \quad (15)$$

where $\boldsymbol{\rho}^{(l)} = \Phi(\mathbf{v}^{(l)})$ for $l = 1, 2$, namely that map Φ preserves the likelihood ratio order.

Note that (15) describes the effect of the initial distribution on the stationary distribution of the restarted process, namely that a larger initial distribution leads to a larger stationary distribution in terms of the likelihood ratio order.

In what follows, we seek a weaker condition to establish a result that is similar to (15), by using the tool of totally positivity functions.

Theorem 2 Suppose $\lambda_k \geq \mu_k$ for $1 \leq k \leq N - 1$. If $\mathbf{v}^{(1)} \leq_{\text{erh}} \mathbf{v}^{(2)}$, then $\boldsymbol{\rho}^{(1)} \leq_{\text{hr}} \boldsymbol{\rho}^{(2)}$.

Proof Denote by $\bar{\rho}_j^{(l)} = \sum_{i=j}^N \rho_i^{(l)}$ the survival function of $\boldsymbol{\rho}^{(l)}$, $l = 1, 2$. It is sufficient to prove that for any $1 \leq i \leq j \leq N$,

$$\frac{\bar{\rho}_i^{(1)} \bar{\rho}_j^{(2)} - \bar{\rho}_i^{(2)} \bar{\rho}_j^{(1)}}{\rho_1^{(1)} \rho_1^{(2)}} \geq 0.$$

Equivalently, to prove that for any $1 \leq i \leq j \leq N$,

$$\sum_{m=i}^N \sum_{k=1}^m \frac{b_{mk}}{\mu_m} \bar{v}_k^{(1)} \cdot \sum_{m=j}^N \sum_{k=1}^m \frac{b_{mk}}{\mu_m} \bar{v}_k^{(2)} - \sum_{m=i}^N \sum_{k=1}^m \frac{b_{mk}}{\mu_m} \bar{v}_k^{(2)} \cdot \sum_{m=j}^N \sum_{k=1}^m \frac{b_{mk}}{\mu_m} \bar{v}_k^{(1)} \geq 0,$$

or

$$\sum_{k=1}^N \sum_{m=i \vee k}^N \frac{b_{mk}}{\mu_m} \bar{v}_k^{(1)} \cdot \sum_{k=1}^N \sum_{m=j \vee k}^N \frac{b_{mk}}{\mu_m} \bar{v}_k^{(2)} - \sum_{k=1}^N \sum_{m=i \vee k}^N \frac{b_{mk}}{\mu_m} \bar{v}_k^{(2)} \cdot \sum_{k=1}^N \sum_{m=j \vee k}^N \frac{b_{mk}}{\mu_m} \bar{v}_k^{(1)} \geq 0, \quad (16)$$

where ' \vee ' represents the maximum, and

$$\bar{v}_k^{(1)} = \sum_{j=k}^N v_j^{(1)}, \quad \bar{v}_k^{(2)} = \sum_{j=k}^N v_j^{(2)}.$$

Let $V_{e,l}$ be the random variable with distribution $\mathbf{v}_e^{(l)}$, $l = 1, 2$. Then inequality (16) can be rewritten as

$$\mathbb{E}[f(i, V_{e,1})] \mathbb{E}[f(j, V_{e,2})] \geq \mathbb{E}[f(j, V_{e,1})] \mathbb{E}[f(i, V_{e,2})]$$

with $1 \leq i \leq j \leq N$, where

$$f(n, k) = \sum_{m=n \vee k}^N \frac{b_{mk}}{\mu_m} = \sum_{m=1}^N \mathbf{I}(m \geq n) \mathbf{I}(m \geq k) \frac{b_{mk}}{\mu_m}.$$

Next, we use Lemma 3 to prove the required result. Regard $V_{e,1}$ as X , $V_{e,2}$ as Y , $f(i, k)$ as $\alpha(k)$, and $f(j, k)$ as $\beta(k)$ in Lemma 3. Then, it is sufficient to verify the following conditions:

- a) $f(j, k)$ is decreasing in k for fixed j ;
- b) $f(j, k)$ is TP_2 in $(j, k) \in \{1, \dots, N\}^2$.

Note that a) follows immediately from the condition $\lambda_k \geq \mu_k$ for $1 \leq k \leq N - 1$. For b), we note that both $\mathbf{I}(m \geq j)$ and $\mathbf{I}(m \geq k)b_{mk}$ are TP_2 in $(m, j) \in \{1, \dots, N\}^2$ and $(m, k) \in \{1, \dots, N\}^2$, respectively. Thus, b) follows from Lemma 2. This completes the proof. ■

Theorem 2 states that when the birth rate is greater than the death rate, the reversed hazard rate order between the equilibrium distributions implies the hazard rate order between the stationary distributions of the restarted process of $\{X(t), t \geq 0\}$. Now we show how to use Theorem 2 to derive the new preservation property.

Theorem 2 and (4) imply the following preservation result.

Corollary 2 Suppose $\lambda_k \geq \mu_k$ for $1 \leq k \leq N - 1$. Then

$$\mathbf{v}^{(1)} \leq_{\text{erh}} \mathbf{v}^{(2)} \implies \boldsymbol{\rho}^{(1)} \leq_{\text{erh}} \boldsymbol{\rho}^{(2)}.$$

5 Further Improved Upper Bound

Now we use the aforementioned equilibrium reversed hazard rate order, which is preserved by Φ , to construct an even better upper bound. Let us first determine the range of ε^* in which $\tilde{\mathbf{q}}_{\varepsilon^*}$ is the upper bound of the QSD of SIS model \mathbf{q} in terms of the equilibrium reversed hazard rate order.

Theorem 3 Suppose $\tau = \lambda/\mu \geq 1$ and $\varepsilon^* \in [\varepsilon_0^*, \infty)$. Then, we have

$$\mathbf{q} \leq_{\text{erh}} \tilde{\mathbf{q}}_{\varepsilon^*},$$

where

$$\varepsilon_0^* = \begin{cases} \frac{1}{N-1}, & \tau \geq \frac{N}{N-1}, \\ (N-1) - (N-2)\tau, & \text{otherwise.} \end{cases} \quad (17)$$

Proof Define the survival functions of $\tilde{\mathbf{q}}_{\varepsilon^*}$ and $\tilde{\mathbf{v}}_{\varepsilon^*}$ as

$$\tilde{q}_{i,\varepsilon^*} = \frac{\sum_{k=i}^N \pi_{k,\varepsilon^*}}{\sum_{k=1}^N \pi_{k,\varepsilon^*}}, \quad \tilde{v}_{i,\varepsilon^*} = \frac{\varepsilon^*}{\pi_{1,\varepsilon^*}} (N-i+1) \pi_{i-1,\varepsilon^*}, \quad i = 1, \dots, N.$$

Similar to the proof of Lemma 5, we first prove

$$\tilde{\mathbf{q}}_{\varepsilon^*} \leq_{\text{erh}} \tilde{\mathbf{v}}_{\varepsilon^*}.$$

According to Definition 2, we need to prove that for any $1 \leq i \leq j \leq N$,

$$\sum_{m=1}^i \sum_{k=m}^N \pi_{k,\varepsilon^*} \cdot \sum_{m=1}^j (N-m+1)\pi_{m-1,\varepsilon^*} \geq \sum_{m=1}^j \sum_{k=m}^N \pi_{k,\varepsilon^*} \cdot \sum_{m=1}^i (N-m+1)\pi_{m-1,\varepsilon^*}, \quad (18)$$

which is equivalent that

$$\frac{\sum_{j=1}^k (N-j+1)\pi_{j-1,\varepsilon^*}}{\sum_{j=1}^k \sum_{i=j}^N \pi_{i,\varepsilon^*}} = \frac{\sum_{j=1}^k (N-j+1)\pi_{j-1,\varepsilon^*}}{\sum_{i=1}^N \pi_{i,\varepsilon^*}(i \wedge k)}$$

is increasing in $1 \leq k \leq N$. Since

$$\frac{\sum_{j=1}^N k\pi_{j,\varepsilon^*}}{\sum_{i=1}^N \pi_{i,\varepsilon^*}(i \wedge k)}$$

is increasing in k , it is sufficient to prove that

$$\frac{1}{k} \sum_{j=1}^k (N-j+1)\pi_{j-1,\varepsilon^*} \quad (19)$$

is increasing in k . Note that a sufficient condition for (19) is that $(N-j)\pi_{j,\varepsilon^*}$ is increasing in $j = 0, 1, \dots, N-1$, i.e., for $j = 1, \dots, N-1$,

$$(N-j)\pi_{j,\varepsilon^*} \geq (N-j+1)\pi_{j-1,\varepsilon^*}. \quad (20)$$

Substituting π_{i,ε^*} into inequality (20) yields

$$(N-j) \binom{N}{j} \prod_{k=0}^{j-1} (\varepsilon^* + k\tau) \geq (N-j+1) \binom{N}{j-1} \prod_{k=0}^{j-2} (\varepsilon^* + k\tau),$$

i.e.,

$$(N-j) \binom{N}{j} (\varepsilon^* + (j-1)\tau) \geq (N-j+1) \binom{N}{j-1}.$$

Simplifying the above inequality yields

$$\varepsilon^* \geq \frac{j}{N-j} - (j-1)\tau.$$

Therefore, a sufficient condition for (19) is

$$\varepsilon^* \geq \varepsilon_0^* = \max_{1 \leq j \leq N-1} \left\{ \frac{j}{N-j} - (j-1)\tau \right\}.$$

Note that

$$h(j) = \frac{j}{N-j} - (j-1)\tau$$

is convex in $j = 1, \dots, N-1$. Hence, the maximum value for $h(j)$ is only possible at $j = 1$ or at $j = N-1$. Therefore, we have

$$\varepsilon_0^* = \begin{cases} \frac{1}{N-1}, & \tau \geq \frac{N}{N-1}, \\ (N-1) - (N-2)\tau, & \text{otherwise.} \end{cases}$$

Now, the inequality (18) holds for any $\varepsilon^* \geq \varepsilon_0^*$. We note that under the condition (20), $\tilde{\mathbf{v}}_{\varepsilon^*}$ is not a distribution because $\tilde{v}_{i,\varepsilon^*} \leq 0$ for $i = 1, \dots, N-1$. However, the inequality (18) is sufficient for the proof of Theorem 2 to remain valid since Lemma 3 is still valid for \mathbf{f} and/or \mathbf{g} including the strictly negative elements. Thus, according to Corollary 2, under the condition $\tau \geq 1$, we have

$$\Phi^n(\tilde{\mathbf{v}}_{\varepsilon^*}) \leq_{\text{erh}} \Phi^{n-1}(\tilde{\mathbf{v}}_{\varepsilon^*}) \leq_{\text{erh}} \cdots \leq_{\text{erh}} \Phi(\tilde{\mathbf{v}}_{\varepsilon^*}) = \tilde{\mathbf{q}}_{\varepsilon^*}.$$

Since

$$\Phi^n(\tilde{\mathbf{v}}_{\varepsilon^*}) \rightarrow \mathbf{q}, \quad n \rightarrow \infty,$$

the required result follows. \blacksquare

Theorem 3 requires the extra condition $\tau \geq 1$, which is however not overly restrictive because it corresponds to the epidemic spreads out, i.e., the extinction time would last a very long time [8].

Now, for $\tau \geq 1$, there are two bounds for the QSD of the SIS model from Theorems 1 and 3, namely

$$\mathbf{q} \leq_{\text{erh}} \tilde{\mathbf{q}}_{\varepsilon_0^*} \quad \text{and} \quad \mathbf{q} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon_+^*}.$$

Since these two orders are stronger than the usual stochastic order, we have

$$\mathbf{q} \leq_{\text{st}} \tilde{\mathbf{q}}_{\varepsilon_0^*} \quad \text{and} \quad \mathbf{q} \leq_{\text{st}} \tilde{\mathbf{q}}_{\varepsilon_+^*}.$$

Moreover, it is easy to note that for a given $\tau > 1$, when N is relative large, it must have $\tau \geq N/(N-1)$ and $\varepsilon_0^* \leq \varepsilon_+^*$. By Lemma 4, we have $\tilde{\mathbf{q}}_{\varepsilon_0^*} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon_+^*}$, and therefore the new upper bound is better than the upper bound given by Theorem 1. From this observation, we obtain the following result.

Corollary 3 For a given $\tau > 1$, it holds that

$$\mathbf{q}_{\varepsilon_0^*} \leq_{\text{lr}} \mathbf{q}_{\varepsilon_+^*} \tag{21}$$

if $N \geq N_0 \triangleq \max\{\frac{1}{\tau-1}, 1 + \frac{1}{\varepsilon_+^*}\}$.

We present a numerical example to illustrate the above result.

Example 2 Consider the SIS model with $\tau = 1.1$. Table 2 presents the numerical values of the QSD \mathbf{q} , $\hat{\mathbf{q}}$, $\mathbf{q}_{\varepsilon_+^*}$ and $\mathbf{q}_{\varepsilon_0^*}$, and compares them in terms of the likelihood ratio order for $N = 15$, where $\varepsilon_+^* = .434$ and $\varepsilon_0^* = .071$ by Eqs. (7) and (17). Note that for $\tau = 1.1 > 1$ and $N = 15 > N_0 = 10$, it follows from Eq. (13), Theorem 3 and Corollary 3 that

$$\mathbf{q} \leq_{\text{erh}} \tilde{\mathbf{q}}_{\varepsilon_0^*} \leq_{\text{lr}} \tilde{\mathbf{q}}_{\varepsilon_+^*} \leq_{\text{lr}} \hat{\mathbf{q}},$$

which coincides with the numerical results presented in Table 2, where we observed that the ratios $q_i/\tilde{q}_{i,\varepsilon_0^*}$, $\tilde{q}_{i,\varepsilon_0^*}/\tilde{q}_{i,\varepsilon_+^*}$ and $\tilde{q}_{i,\varepsilon_+^*}/\hat{q}_i$ are decreasing in $i = 1, \dots, 15$.

Similar to Example 1, in order to understand the effect of population sizes, Figure 2 further displays the fractions of infected individuals of different population sizes in the quasi-stationary state, and their upper bounds from $\tilde{\mathbf{q}}_{\varepsilon_0^*}$, $\tilde{\mathbf{q}}_{\varepsilon_+^*}$ and $\hat{\mathbf{q}}$. It is seen that the bound from $\tilde{\mathbf{q}}_{\varepsilon_0^*}$ is the best one among those upper bounds as expected. \blacksquare

i	q_i	$\tilde{q}_{\varepsilon_0^*,i}$	$\tilde{q}_{\varepsilon_+^*,i}$	\hat{q}_i	$q_i/\tilde{q}_{i,\varepsilon_0^*}$	$\tilde{q}_{i,\varepsilon_0^*}/\tilde{q}_{i,\varepsilon_+^*}$	$\tilde{q}_{i,\varepsilon_+^*}/\hat{q}_i$
1	1.595×10^{-11}	1.392×10^{-11}	5.391×10^{-12}	1.217×10^{-12}	1.146	2.582	4.430
2	1.308×10^{-10}	1.142×10^{-10}	5.790×10^{-11}	1.874×10^{-11}	1.145	1.972	3.090
3	1.252×10^{-9}	1.124×10^{-9}	6.609×10^{-10}	2.680×10^{-10}	1.114	1.701	2.466
4	1.240×10^{-8}	1.137×10^{-8}	7.403×10^{-9}	3.538×10^{-9}	1.091	1.536	2.093
5	1.200×10^{-7}	1.118×10^{-7}	7.873×10^{-8}	4.281×10^{-8}	1.073	1.420	1.839
6	1.100×10^{-6}	1.038×10^{-6}	7.787×10^{-7}	4.709×10^{-7}	1.060	1.333	1.654
7	9.337×10^{-6}	8.907×10^{-6}	7.042×10^{-6}	4.661×10^{-6}	1.048	1.265	1.511
8	7.190×10^{-5}	6.922×10^{-5}	5.728×10^{-5}	4.102×10^{-5}	1.039	1.208	1.396
9	4.921×10^{-4}	4.776×10^{-4}	4.114×10^{-4}	3.159×10^{-4}	1.030	1.161	1.302
10	2.923×10^{-3}	2.857×10^{-3}	2.551×10^{-3}	2.085×10^{-3}	1.023	1.120	1.224
11	1.461×10^{-2}	1.438×10^{-2}	1.326×10^{-2}	1.147×10^{-2}	1.016	1.084	1.156
12	5.894×10^{-2}	5.834×10^{-2}	5.539×10^{-2}	5.045×10^{-2}	1.010	1.053	1.098
13	1.796×10^{-1}	1.787×10^{-1}	1.743×10^{-1}	1.665×10^{-1}	1.005	1.025	1.047
14	3.668×10^{-1}	3.668×10^{-1}	3.668×10^{-1}	3.663×10^{-1}	1.000	1.000	1.001
15	3.766×10^{-1}	3.784×10^{-1}	3.872×10^{-1}	4.029×10^{-1}	.995	.977	.961

Table 2: Comparing the numerical result of QSD q , the newly obtained upper bounds $\tilde{q}_{\varepsilon_0^*}$, $\tilde{q}_{\varepsilon_+^*}$ and the state-of-the-art upper bound \hat{q} in the SIS model, where $\tau = 1.1$ and $N = 15$.

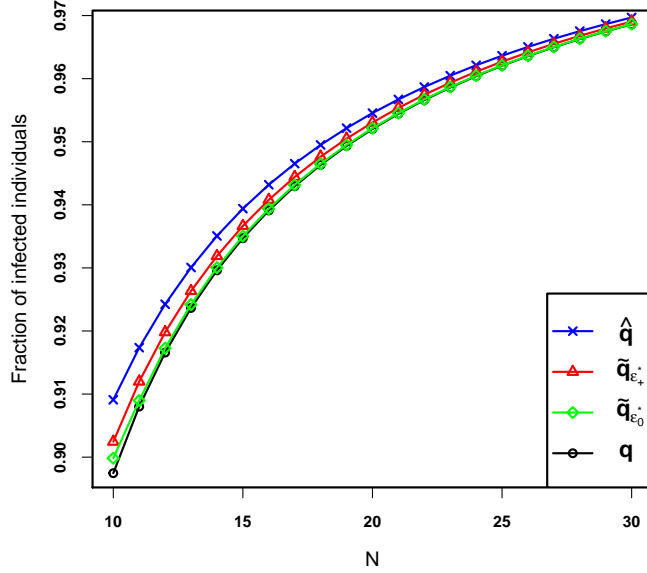


Figure 2: The fractions of infected individuals of the populations with different sizes in the quasi-stationary state and their upper bounds corresponding to \hat{q} , $\tilde{q}_{\varepsilon_+^*}$ and $\tilde{q}_{\varepsilon_0^*}$ respectively, where $\tau = 1.1$.

6 Concluding Remarks

We have presented a novel method for constructing the upper bound for the QSD of the SIS model. The new upper bound is better than the one given in the literature in terms of the likelihood ratio order. We also have showed that the widely used map Φ preserves the equilibrium reversed hazard rate order under some suitable condition. We have used this new preservation property to further improve the new upper bound for the QSD of the SIS model.

It should be mentioned that although the new upper bounds $\mathbf{q}_{\varepsilon_+^*}$ and $\mathbf{q}_{\varepsilon_0^*}$ are better than the state-of-the-art upper bound $\hat{\mathbf{q}}$, they are not necessarily the best ones. The reason is that the upper bounds $\mathbf{q}_{\varepsilon_+^*}$ and $\mathbf{q}_{\varepsilon_0^*}$ are obtained in terms of the likelihood ratio order and the equilibrium reversed hazard rate order, respectively. Both of them are stronger than the usual stochastic order. Further, the conditions presented in Lemma 5 and Theorem 3 are sufficient but not necessary conditions. As pointed out by the referee, there may exist a smaller $\varepsilon^* < \min\{\varepsilon_+^*, \varepsilon_0^*\}$ such that $\mathbf{q} \leq_{\text{st}} \mathbf{q}_{\varepsilon^*}$. The following numerical results confirm this fact as well.

Consider a SIS model with $N = 10$.

- (1) For $\tau = 0.15$, we have $\varepsilon_+^* = 0.118$. Setting $\varepsilon_1^* = 0.08$, it is seen from Table 3 that $\mathbf{q} \leq_{\text{st}} \mathbf{q}_{\varepsilon_1^*}$.

i	1	2	3	4	5	6	7	8	9	10
\bar{q}_i	1	.827	.640	.453	.286	.156	.070	.024	.006	.001
$\tilde{q}_{\varepsilon_1^*, i}$	1	.830	.654	.475	.310	.175	.082	.029	.007	.001

Table 3: Comparing survival functions of QSD \mathbf{q} and upper bound $\mathbf{q}_{\varepsilon_1^*}$ with $\tau = 0.15$, $\varepsilon_1^* = 0.08$, and $\varepsilon_+^* = 0.118$. \bar{q}_i represents the survival function of QSD, and $\tilde{q}_{\varepsilon_1^*, i}$ represents the survival function of $\mathbf{q}_{\varepsilon_1^*}$.

- (2) For $\tau = 1.1$, we have $\varepsilon_0^* = 0.2$ and $\varepsilon_+^* = 0.434$. Setting $\varepsilon_2^* = 0.1$, we observe from Table 4 that $\mathbf{q} \leq_{\text{st}} \mathbf{q}_{\varepsilon_2^*}$.

i	1	2	3	4	5	6	7	8	9	10
\bar{q}_i	1	1	1	1	.999	.995	.978	.914	.727	.362
$\tilde{q}_{\varepsilon_2^*, i}$	1	1	1	1	.999	.996	.979	.916	.731	.366

Table 4: Comparing survival functions of QSD \mathbf{q} and upper bound $\mathbf{q}_{\varepsilon_2^*}$ with $\tau = 1.1$, $\varepsilon_2^* = 0.1$, $\varepsilon_0^* = 0.2$, and $\varepsilon_+^* = 0.434$. \bar{q}_i represents the survival function of QSD, and $\tilde{q}_{\varepsilon_2^*, i}$ represents the survival function of $\mathbf{q}_{\varepsilon_2^*}$.

One may wonder whether we could use an arbitrary small ε^* to bound the QSD. The answer is negative. For example, let $\tau = 0.15$, $\varepsilon_3^* = 0.05$, and $\varepsilon_+^* = 0.118$. Then, one could calculate that

$$\bar{q}_4 = .453 > \tilde{q}_{\varepsilon_3^*, 4} = .423.$$

Therefore, $\varepsilon_3^* = 0.05$ can not be used to construct the upper bound for the QSD. This also raises an interesting question about the critical value of ε^* for the upper bound of QSD. This problem is currently open, and left for the future study.

There are also many interesting problems that are left for future research. First, it is interesting to study the QSD of the SIS model while accommodating heterogeneous network topologies, such as star graphs, random graphs, and power-law graphs. Second, it may be possible to derive some new asymptotic

approximations for the QSD of the SIS model. Third, it is interesting to construct variance bounds for the QSD of the SIS model. For example, how can we bound the variance of the QSD of the SIS model?

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